Introduction

Welcome to Math 125

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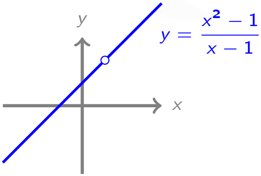
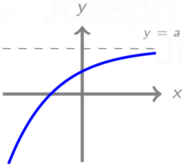
Introduction to Calculus

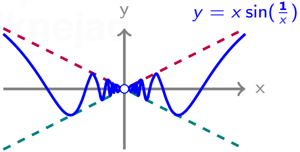
Calculus is the study of **change**. That is,
(1) study of change when elapsed time is extremely small (infinitesimal).
(2) calculating the behavior of mathematical models.

Limits and Continuity (Chapter 2)

Limits and continuity are essential in deep understanding of the infinitesimal change and finding the behavior of models.

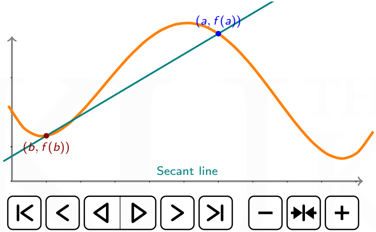




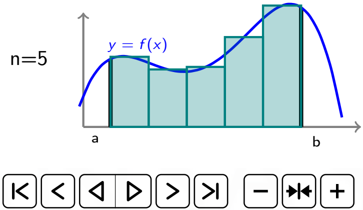


Some Concepts in Calculus I

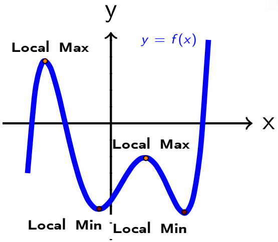
Rate of Change (Chapter 3)



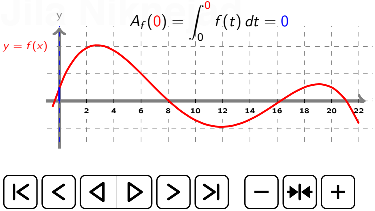
Area Under Curves (Chapter 5)



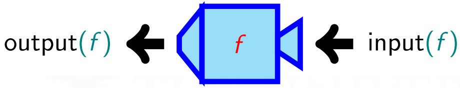
Optimization (Chapter 4)



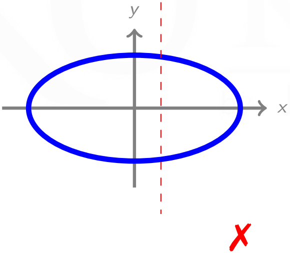
Fundamental Theorem

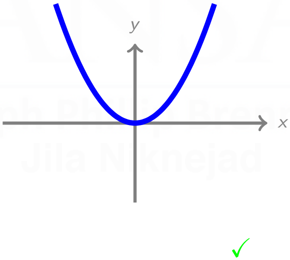


Function as a Black Box

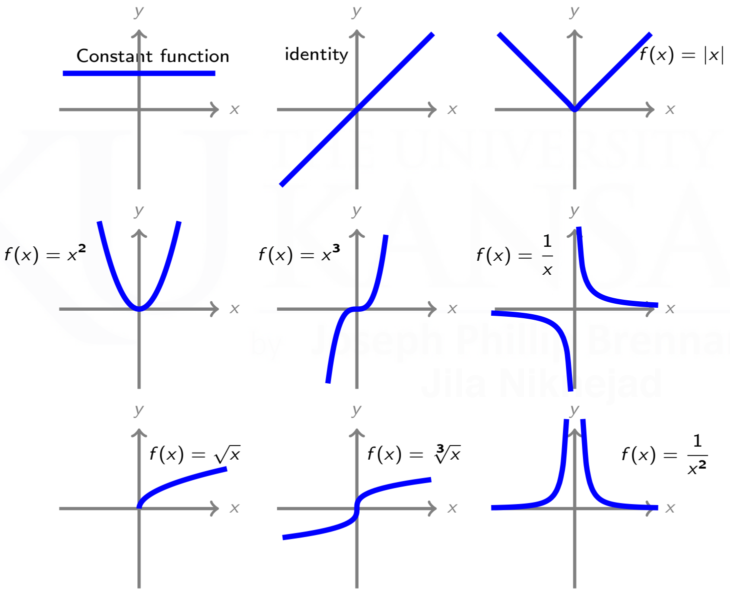


Vertical Line Test

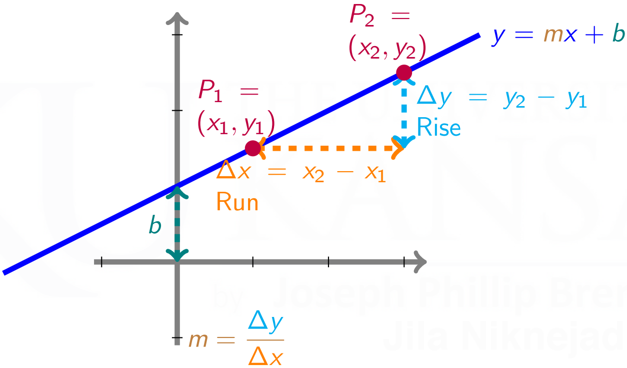




Graphs of Well-known Functions

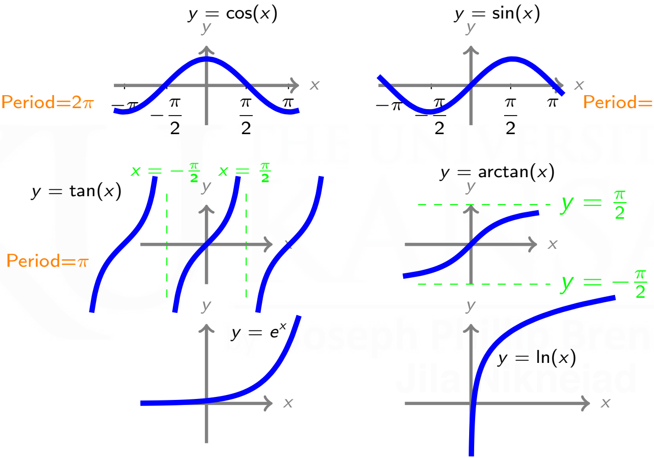


Equation of a Line



- $y = mx + b$ is called the **slope intercept form.** $y = \underbrace{m}_{\text{Slope}} x + \underbrace{b}_{\text{y-intercept}}$
- Graph of a linear function is a line.

Trig, Exponential and Logarithmic Graphs



What are the Horizontal and Vertical Asymptotes of these graphs? What are the symmetries? How are the graphs of inverse functions related?

Example, Domain and Composition of Functions

- Composition and domain $f(x) = \sqrt{x}, g(x) = \dfrac{x+1}{x-2}$.

  Domain of $f(x) = \sqrt{x}$ is $[0, \infty)$.

  Domain of $g(x) = \dfrac{x+1}{x-2}$ is $(-\infty, 2) \cup (2, \infty)$.

  $(f \circ g)(x) = \sqrt{\dfrac{x+1}{x-2}}$ and $(g \circ f)(x) = \dfrac{\sqrt{x}+1}{\sqrt{x}-2}$

  Domain of $(g \circ f)$ is $[0, 4) \cup (4, \infty)$.

Laws of Exponents and Logarithms

Laws of Exponents

Let a and b be positive numbers and let x and y be real numbers. Then,

1. $b^x . b^y = b^{x+y}$
2. $\dfrac{b^x}{b^y} = b^{x-y}$
3. $(b^x)^y = b^{xy}$
4. $(ab)^x = a^x b^x$
5. $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$

Laws of Logarithms

If m and n are positive numbers and $b > 0$, $b \neq 1$, then

1. $\log_b(mn) = \log_b(m) + \log_b(n)$
2. $\log_b\left(\frac{m}{n}\right) = \log_b(m) - \log_b(n)$
3. $\log_b(m^n) = n\log_b(m)$
4. $\log_b(1) = 0$
5. $\log_b(b) = 1$

Examples, Laws of Exponents and Logarithms

- Evaluate: **(1)** $\log_3\left(\frac{1}{27}\right)$ (1) $\log_5(125)$

(1) $\log_3\left(\dfrac{1}{27}\right) = \log_3(1) - \log_3(27) = 0 - \log_3\left(3^3\right) = -3\log_3(3) = -3$

(2) $\log_5(125) = \log_5\left(5^3\right) = 3\log_5(5) = 3$

- Solve: **(1)** $\ln(x) + \ln(x-1) = 1$ **(2)** $e^{7-4x} = 6$

(1) $\ln(x) + \ln(x-1) = 1 \implies \ln(x(x-1)) = 1 \implies x(x-1) = e \implies x^2 - x - e = 0 \underset{\text{Quadratic formula}}{\implies}$

$$x = \frac{1 \pm \sqrt{1+4e}}{2} \begin{cases} \nearrow & x = \dfrac{1-\sqrt{1+4e}}{2} < 0 \quad \text{Not in the domain ✗} \\ \searrow & \dfrac{1+\sqrt{1+4e}}{2} \text{ ✓} \end{cases}$$

(2) $e^{7-4x} = 6 \implies 7 - 4x = \ln(6) \implies x = \dfrac{7-\ln(6)}{4}$

Examples

- Simplify: $\frac{x^{-2}y\sqrt{z^5}}{x\sqrt{y^4z}} = \frac{yz^{5/2}}{x^2xy^{4/2}z^{1/2}} = \frac{z^2}{x^3y}$
- Factoring and Simplification

$$\text{Factor: } 18(x+2)^{-2}(x-1)^{\frac{1}{2}} - 6(x+2)^{-1}(x-1)^{\frac{3}{2}}$$

Factor each common factor with the lowest exponent:

$$\underbrace{(2)(3)(x+2)^{-2}(x-1)^{1/2}}_{\text{common factor}}\Big(3-(x+2)(x-1)\Big) =$$

$$6\underbrace{(x+2)^{-2}}_{\text{Convert}}(x-1)^{1/2}(-x^2-x+5) = \frac{6\sqrt{x-1}(-x^2-x+5)}{(x+2)^2}$$

Graph of Functions and Horizontal Asymptotes

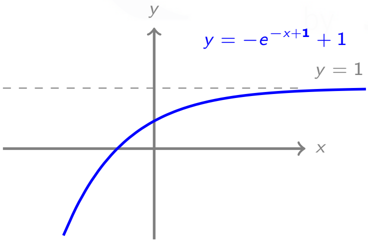
Horizontal asymptotes indicate the stable end behavior of functions. Many elementary functions that we are familiar with have horizontal asymptotes and a review of all of those are necessary.

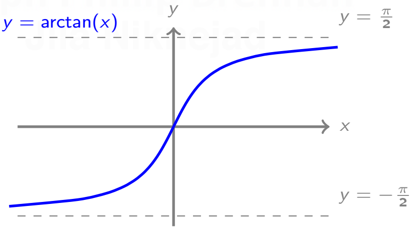
Example:

Graph the following function indicating its asymptotes.

$y = -e^{-x+1} + 1$

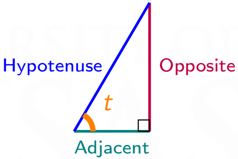
$y = \arctan(x)$





Trigonometric Ratios

Hypotenuse is the side opposite of the right angle.
Adjacent is the other side of the angle x.
Opposite is the side opposite of angle x.



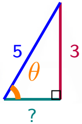
$$\sin(t) = \frac{opposite}{hypotenuse} \qquad \tan(t) = \frac{opposite}{adjacent} \qquad \sec(t) = \frac{hypotenuse}{adjacent}$$

$$\cos(t) = \frac{adjacent}{hypotenuse} \qquad \cot(t) = \frac{adjacent}{opposite} \qquad \csc(t) = \frac{hypotenuse}{opposite}$$

Examples, Trigonometric Ratios

- $\frac{\sin(x)}{x} \neq \sin \qquad \sin(x+y) \neq \sin(x) + \sin(y)$

- Find other trig functions if $\sin(\theta) = \dfrac{3}{5}, 0 < \theta < \dfrac{\pi}{2}$.



$$\sin(\theta) = \frac{3}{5} \implies \cos(\theta) = \frac{4}{5},\ \tan(\theta) = \frac{3}{4},\ \sec(\theta) = \frac{5}{4}$$

Example, Piecewise-defined

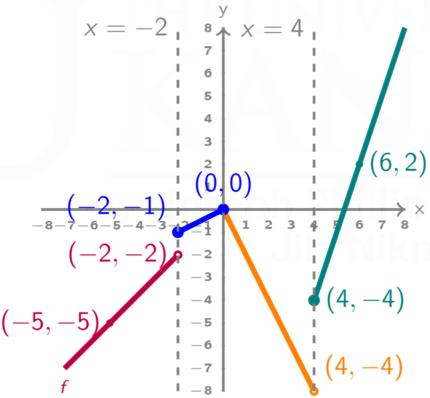
Graph

$$f(x) = \begin{cases} x & x < -2 \\ 0.5x & -2 \leq x \leq 0 \\ -2x & 0 < x < 4 \\ 3x - 16 & x \geq 4 \end{cases}$$

Label two points of each linear piece of the graph.

Solution:

General Process: (1)Divide the domain into the pieces according to the conditions. (2)Graph each piece in the corresponding region.
For this example, remember that any line can be drawn using two points.



Absolute Value Functions

- Absolute Value functions are piecewise-defined functions.

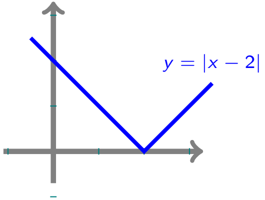
$$|x| = \begin{cases} x & \text{when } x \geq 0 \\ -x & \text{when } x < 0 \end{cases}$$

(Check this fact by taking a sample value in each rule.)

- Rewrite the absolute value function as a piecewise-defined function:

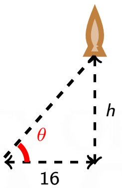
$$f(x) = |x-2| = \begin{cases} x-2 & \text{when } x-2 \geq 0 \\ -(x-2) & \text{when } x-2 < 0 \end{cases} \Longrightarrow$$

$$|x-2| = \begin{cases} x-2 & x \geq 2 \\ -x+2 & x < 2 \end{cases}$$



Examples, Modeling with Mathematics

An observer is viewing the space shuttle take off from a distance of 16 kilometers from the launch pad. The shuttle travels straight up with constant speed of 8 kilometers per second for the first part of the ascent. Let t be the time passed, in seconds, since the shuttle has been launched. Let θ be the angle of elevation in radians.



(1) Express the the vertical distance of the shuttle, h, as a a function of time, t, in seconds. $\boxed{h(t) = 8t}$

(2) Express the angle of elevation θ as a function of the vertical distance of the shuttle, h, in kilometers. $\tan(\theta) = \frac{h}{16} \implies \boxed{\theta(h) = \arctan\left(\frac{h}{16}\right)}$

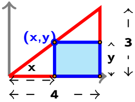
(3) Express the angle of elevation, θ, as a function of time, t, in seconds. $\boxed{\theta(h(t)) = \arctan\left(\frac{8t}{16}\right)}$

(4) What is the vertical distance, in kilometers, when $t = 2$ seconds.
$h(2) = \boxed{16}$

(5) What is the angle of elevation, in radians, when $t = 2$ seconds.
$\theta = \arctan\left(\frac{8(2)}{16}\right) = \boxed{\frac{\pi}{4}}$

Examples, Modeling, Optimization

What is the **maximum area** of a rectangle inscribed in a right triangle with side lengths 3 and 4, if the sides of the rectangle are parallel to the legs of the triangle?



Link

- Constraint: The triangle. By similarity of triangles, $\dfrac{y}{x} = \dfrac{3}{4} = 0.75$.
- Express y as a function of x: $y = 0.75x$.
- Objective function: Area is $A = y(4-x)$. $\Longrightarrow$
  $A(x) = 0.75x(4-x) \underset{\text{Distribute}}{\Longrightarrow} A(x) = 0.75(4x - x^2)$. (a parabola)
- Domain: $(0, 4)$.
- Is the vertex a maximum? Yes, the function is a downward parabola.
- x-coordinate of the vertex: $x = \dfrac{-4}{2(-1)} = 2$.
- The maximum area by plugging into the objective function:
  $A(2) = 0.75(4 \times 2 - 2^2) = \boxed{3}$.